A new McKean-Vlasov stochastic interpretation of the parabolic-parabolic Keller-Segel model: The one-dimensional case

Milica Tomasevic, Denis Talay

A new McKean-Vlasov stochastic interpretation of the parabolic-parabolic Keller-Segel model: The one-dimensional case.

Denis Talay*and Milica Tomašević *†

Abstract: In this paper we analyze a stochastic interpretation of the one-dimensional parabolic-parabolic Keller-Segel system without cut-off. It involves an original type of McKean-Vlasov interaction kernel. At the particle level, each particle interacts with all the past of each other particle by means of a time integrated functional involving a singular kernel. At the mean-field level studied here, the McKean-Vlasov limit process interacts with all the past time marginals of its probability distribution in a similarly singular way. We prove that the parabolic-parabolic Keller-Segel system in the whole Euclidean space and the corresponding McKean-Vlasov stochastic differential equation are well-posed for any values of the parameters of the model.
Key words: Chemotaxis model; Keller–Segel system; Singular McKean-Vlasov non-linear stochastic differential equation.
Classification: 60H30 60H10 60K35.

1 Introduction

The standard d-dimensional parabolic–parabolic Keller–Segel model for chemotaxis describes the time evolution of the density ρ_t of a cell population and of the concentration c_t of a chemical attractant:

$$\begin{cases} \partial_t \rho(t,x) = \nabla\cdot(\frac{1}{2}\nabla\rho - \chi\rho\nabla c)(t,x), \quad t>0,\ x\in\mathbb{R}^d, \\ \alpha\ \partial_t c(t,x) = \frac{1}{2}\triangle c(t,x) - \lambda c(t,x) + \rho(t,x), \quad t>0,\ x\in\mathbb{R}^d. \\ \rho(0,x)=\rho_0(x), \quad c(0,x)=c_0(x), \end{cases} \tag{1}$$

where $\chi,\alpha>0$ and $\lambda\geq 0$. See e.g. [2], [12] and references therein for theoretical results on this system of PDEs and applications to Biology.

Recently, stochastic interpretations have been proposed for a simplified version of the two-dimensional model, that is, the parabolic-elliptic model which corresponds to the values $\alpha=0$ and $\lambda=0$. They all rely on the fact that, in the parabolic-elliptic case, $\nabla c(t,x)$ can be explicited as the convolution of $\rho(t,x)$ and the kernel $k(x)=-\frac{x}{2\pi|x|^2}$, which allows one to rewrite the first equation in (1) as a closed standard McKean–Vlasov–Fokker–Planck equation.

Consequently, the stochastic process of McKean–Vlasov type whose ρ_t is the time marginal density involves the singular interaction kernel k. This explains why, so far, only partial results are obtained and heavy techniques are used to get them. In [8], one may find a short review of the works [6], [5] and [3].

We here deal with the parabolic–parabolic system ($\alpha=1$) without cut-off and study the McKean-Vlasov stochastic representation of the mild formulation of the equation satisfied by ρ_t. This representation involves a singular interaction kernel which is different from the one in the above mentioned approaches and does not seem to have been studied in the McKean-Vlasov non-linear SDE literature. The system reads

$$\begin{cases} dX_t = b^\sharp(t,X_t)dt + \left\{\int_0^t (K^\sharp_{t-s}*p_s)(X_t)ds\right\}dt + dW_t, \quad t>0, \\ p_s(y)dy := \mathcal{L}(X_s), \quad s>0, \end{cases} \tag{2}$$

where $K^\sharp_t(x) := \chi e^{-\lambda t}\nabla(\frac{1}{(2\pi t)^{d/2}}e^{-\frac{|x|^2}{2t}})$ and $b^\sharp(t,x) := \chi e^{-\lambda t}\nabla\mathbb{E}c_0(x+W_t)$. Here, $(W_t)_{t\geq 0}$ is a d-dimensional Brownian motion on a filtered probability space $(\Omega,\mathcal{F},\mathbb{P},(\mathcal{F}_t))$ and X_0 is an $\mathbb{R}^d$-valued $\mathcal{F}_0-$measurable random variable. Notice that the formulation requires that the one dimensional time marginals of the law of

*TOSCA, Inria Sophia Antipolis-Méditerranée, 2004 Route des Lucioles, 06902 Valbonne, France (denis.talay@inria.fr, milica.tomasevic@inria.fr).
†Université Côte d'Azur, Campus Valrose, Batiment M, 28 Avenue de Valrose, 06108 Nice CEDEX 2, France.

the solution are absolutely continuous with respect to Lebesgue's measure and that the process interacts with all the past time marginals of its probability distribution through a functional involving a singular kernel.

For $d = 2$ and χ small enough, the analysis of the well-posedness of this non-linear equation and the proof that $(p_s)_{s\geq 0}$, together with some well chosen $(c_s)_{s\geq 0}$, solves the Keller-Segel equation can be found in [14]. For larger values of χ, these issues become delicate because solutions may blow up in finite time. As numerical simulations of the related particle system appear to be effective for arbitrary value of χ, it seems interesting to validate our probabilistic approach in the one-dimensional case.

The objective of this paper is to prove general existence and uniqueness results for both the deterministic system (1) and the stochastic dynamics (2) for $d = 1$, $\alpha = 1$ and any $\chi > 0$. The companion paper [8] deals with the well-posedness and propagation of chaos property of the particle system corresponding to (2) for $d = 1$. There, each particle interacts with all the past of all the other ones by means of a time integrated singular kernel.

In this one-dimensional framework, the PDE (1) was previously studied by [11, 7] in bounded intervals I with boundary conditions while we here deal with the problem posed on the whole space $\mathbb{R}$. In [11] one assumes that the density ρ_0 is in $L^2(I)$ and c_0 is in $H^1(I)$. In [7] one assumes $\rho_0 \in L^\infty(I)$ and $c_0 \in W^{q,r}(I)$, where q and r belong to a particular set of parameters. Here, we only suppose that ρ_0 is a probability measure (not necessarily a density function) and $c_0 \in C_b^1(\mathbb{R})$.

We emphasize that we do not limit ourselves to the specific kernel $K_t^\sharp(x)$ related to the Keller–Segel model. We below show that the mean–field PDE and the stochastic differential equation of Keller-Segel type are well-posed for a whole class of time integrated singular kernels. This SDE cannot be analyzed by means of standard coupling methods or Wasserstein distance contractions. Both to construct local solutions and to go from local to global solutions, an important issue consists in properly defining the set of weak solutions without any assumption on the initial probability distribution of X_0. That led us to introduce constraints on the time marginal densities. To prove that these constraints are satisfied in the limit of an iterative procedure where the kernel is not cut off, the norms of the successive time marginal densities cannot be allowed to exponentially depend on the L^∞-norm of the successive corresponding drifts. They neither can be allowed to depend on Hölder-norms of the drifts. Therefore, we use an accurate estimate (with explicit constants) on densities of one-dimensional diffusions with bounded measurable drifts which is obtained by a stochastic technique rather than by PDE techniques. This strategy allows us to get uniform bounds on the L^∞-norms of the sequence of drifts, which is essential to get existence and uniqueness of the local solution to the non-linear martingale problem solved by any limit of the Picard procedure, and to suitably paste local solutions when constructing the global solution.

The paper is organized as follows. In Section 2 we state our main results. In Section 3 we prove a preliminary estimate on the probability density of diffusions whose drift is only supposed Borel measurable and bounded. In Section 4 we study a non-linear McKean-Vlasov-Fokker-Planck equation. In Section 5 we prove the local existence and uniqueness of a solution to a non-linear stochastic differential equation more general than (2) (for $d = 1$). In Section 6 we get the global well-posedness of this equation. In Section 7 we apply the preceding result to the specific case of the one-dimensional parabolic–parabolic Keller-Segel model. The appendix section 8 concerns an explicit formula for the transition density of a particular diffusion.

Notation. In all the paper we denote by C_T, $C_T(b_0, p_0)$, etc., any constant which depends on T and the other specified parameters, but is uniform w.r.t. $t \in [0, T]$ and may change from line to line. Similarly, C denotes any universal constant which may change from line to line.

2 Our main results

Our first main result concerns the well-posedness of a non-linear one-dimensional stochastic differential equation with a non standard McKean–Vlasov interaction kernel which at each time t involves in a singular way all the time marginals up to time t of the probability distribution of the solution. As our technique of analysis is not limited to the above kernel $K^\sharp$, we consider the following McKean-Vlasov stochastic equation:

$$\begin{cases} dX_t = b(t, X_t)dt + \left\{ \int_0^t (K_{t-s} * p_s)(X_t)ds \right\} dt + dW_t, \quad t \leq T, \\ p_s(y)dy := \mathcal{L}(X_s), \ s > 0; \quad X_0 \sim p_0(dy), \end{cases} \tag{3}$$

and in all the sequel we assume the following conditions on the interaction kernel.

Hypothesis (H)**.** *The function K defined on $\mathbb{R}^+ \times \mathbb{R}$ is such that*

1. *For any $t > 0$, K_t is in $L^1(\mathbb{R})$.*
2. *For any $t > 0$ the function $K_t(x)$ is a bounded continuous function on $\mathbb{R}$.*
3. *The set of points $x \in \mathbb{R}$ such that $\lim_{t\to 0} K_t(x) < \infty$ has full Lebesgue measure.*
4. *For any $T > 0$ the function $f_1(t) := \int_0^t \frac{\|K_{t-s}\|_{L^1(\mathbb{R})}}{\sqrt{s}} ds$ is well defined and bounded on $[0, T]$.*
5. *For any $T > 0$ there exists C_T such that*
$$\sup_{x\in\mathbb{R}} \|K_\cdot(x)\|_{L^1(0,T)} \leq C_T.$$
6. *For any $T > 0$ there exists C_T such that*
$$\sup_{0\leq t\leq T} \int_0^T \|K_{T+t-s}\|_{L^1(\mathbb{R})} \frac{1}{\sqrt{s}} ds \leq C_T.$$

As emphasized in the introduction, the well-posedness of the system (3) cannot be obtained by applying known results in the literature.

Given $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$ and a family of densities $(p_t)_{t\leq T}$ we set

$$B(t, x; p) := \int_0^t (K_{t-s} * p_s)(x) ds. \tag{4}$$

We now define the notion of a weak solution to (3).

Definition 2.1. *The family $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), X, W)$ is said to be a weak solution to the equation* (3) *up to time $T > 0$ if:*

1. *$(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t))$ is a filtered probability space.*
2. *The process $W := (W_t)_{t\in[0,T]}$ is a one-dimensional $(\mathcal{F}_t)$-Brownian motion.*
3. *The process $X := (X_t)_{t\in[0,T]}$ is real-valued, continuous, and $(\mathcal{F}_t)$-adapted. In addition, the probability distribution of X_0 is p_0.*
4. *The probability distribution $\mathbb{P}\circ X^{-1}$ has time marginal densities $(p_t,\ t \in (0, T])$ with respect to Lebesgue measure which satisfy*
$$\forall 0 < t \leq T, \quad \|p_t\|_{L^\infty(\mathbb{R})} \leq \frac{C_T}{\sqrt{t}}. \tag{5}$$
5. *For any $t \in (0, T]$ and $x \in \mathbb{R}$, one has that $\int_0^t |b(s, X_s)|\, ds < \infty$ a.s.*
6. *$\mathbb{P}$-a.s. the pair (X, W) satisfies* (3)*.*

Remark 2.2. *For any $T > 0$, Inequality* (5) *and Hypothesis (H-4) lead to*

$$\exists C_T > 0, \quad \sup_{0\leq t\leq T} \sup_{x\in\mathbb{R}} |B(t, x; p)| \leq C_T,$$

which means that the the drift term in (3) *is bounded.*

The following theorem provides existence and uniqueness of the weak solution to (3).

Theorem 2.3. *Let $T > 0$. Suppose that $b \in L^\infty([0, T] \times \mathbb{R})$ is continuous w.r.t. the space variable. Under the hypothesis (H), Eq.* (3) *admits a unique weak solution in the sense of Definition 2.1.*

We finally state an easy result which is useful to prove the propagation of chaos in the case of Keller-Segel kernel (see [8]):

Corollary 2.4. *Let $r \in (1,\infty)$ and r' such that $\frac{1}{r}+\frac{1}{r'}=1$. In addition to the assumptions of Theorem 2.3 suppose the following hypothesis:*

H-7. For any $t>0$, K_t is in $L^{r'}(\mathbb{R})$ and the function $f_2(t) := \int_0^t \frac{\|K_{t-s}\|_{L^{r'}(\mathbb{R})}}{s^{\frac{1}{2r'}}} ds$ is well defined and bounded on $[0,T]$.

Then, Definition 2.1 is equivalent to Definition 2.1 modified as follows: Instead of (5) *one imposes*

$$\forall 0 < t \le T, \quad \|p_t\|_{L^r(\mathbb{R})} \le \frac{C_T}{t^{\frac{1}{2}(1-\frac{1}{r})}}. \tag{6}$$

Our next result concerns the well-posedness of the the one-dimensional parabolic-parabolic Keller-Segel model

$$\begin{cases} \dfrac{\partial \rho}{\partial t}(t,x) = \dfrac{\partial}{\partial x}\cdot(\dfrac{1}{2}\dfrac{\partial \rho}{\partial x} - \chi\rho\dfrac{\partial c}{\partial x})(t,x), \quad t>0, \quad x\in\mathbb{R}, & \text{(7a)}\\ \dfrac{\partial c}{\partial t}(t,x) = \dfrac{1}{2}\dfrac{\partial^2 c}{\partial x^2}(t,x) - \lambda c(t,x) + \rho(t,x), \quad t>0, \quad x\in\mathbb{R}, & \text{(7b)}\\ \rho(t,\cdot) \xrightarrow{w} \rho_0(dx), \ t\to 0; \quad c(0,x) = c_0(x). \end{cases}$$

As this system preserves the total mass, that is,

$$\forall t>0, \qquad \int_\Omega \rho(t,x)dx = \int_\Omega \rho_0(dx) =: M,$$

the new functions $\tilde\rho(t,x) := \frac{\rho(t,x)}{M}$ and $\tilde c(t,x) := \frac{c(t,x)}{M}$ satisfy the system (7) with the new parameter $\tilde\chi := \chi M$. Therefore, w.l.o.g. we may and do thereafter assume that $M=1$.

Denote $g_t(x) := \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$. We define the notion of solution for the system (7):

Definition 2.5. *Given the probability measure ρ_0, the function c_0, and the constants $\chi>0$, $\lambda\ge 0$, $T>0$, the pair (ρ,c) is said to be a solution to* (7) *if for every $0<t\le T$ the function $\rho(t,\cdot)$ is a probability density function which satisfies $\|\rho(t,\cdot)\|_{L^\infty(\mathbb{R})} \le \frac{C_T}{\sqrt{t}}$, c is in $L^\infty([0,T];C_b^1(\mathbb{R}))$, and the following equality*

$$\rho(t,x) = g_t * \rho_0(x) - \chi\int_0^t \frac{\partial g_{t-s}}{\partial x} * (\frac{\partial c}{\partial x}(s,\cdot)\ \rho(s,\cdot))(x)\ ds \tag{8}$$

is satisfied in the sense of the distributions with

$$c(t,x) = e^{-\lambda t}(g(t,\cdot\,) * c_0)(x) + \int_0^t e^{-\lambda s}(g_s * \rho(t-s,\cdot))(x)\ ds. \tag{9}$$

Notice that the function $c(t,x)$ defined by (9) is a mild solution to (7b). These solutions are known as integral solutions and they have already been studied in PDE literature for the two-dimensional Keller-Segel model for which sub-critical and critical regimes exist depending on the parameters of the model (see [4] and references therein). In the one-dimensional case there is no critical regime as shown by the following theorem.

Theorem 2.6. *Assume that ρ_0 is a probability measure and $c_0\in C_b^1(\mathbb{R})$. Given any $\chi>0$, $\lambda\ge 0$ and $T>0$, for $t>0$ consider the time marginal densities $\rho(t,x)\equiv p_t(x)$ of the probability distribution of the unique weak solution to Eq.* (3) *with $K=K^\sharp$ and $b=b^\sharp$. Also consider the corresponding function $c(t,x)$ as in* (9)*. The pair (ρ,c) provides a global solution to* (7) *in the sense of Definition 2.5. Any other solution (ρ^1,c^1) with the same initial condition (ρ_0,c_0) satisfies $\|\rho^1(t,\cdot)-\rho(t,\cdot)\|_{L^1(\mathbb{R})}=0$ and $\|\frac{\partial c^1}{\partial x}(t,\cdot)-\frac{\partial c}{\partial x}(t,\cdot)\|_{L^1(\mathbb{R})}=0$ for every $0<t\le T$.*

Remark 2.7. *From estimates below we could deduce some additional regularity results which we do not need here: See Remark 3.3. In particular, if the initial condition has a density $\rho_0\in L^\infty(\mathbb{R})$, then $\rho\in L^\infty([0,T];L^1\cap L^\infty(\mathbb{R}))$. If $\rho_0\in L^2(\mathbb{R})$, then $\rho\in L^\infty([0,T];L^1\cap L^2(\mathbb{R}))$ and $t^{1/4}\|\rho_t\|_{L^\infty(\mathbb{R})}\le C$.*

Remark 2.8. *In our definitions 2.1 and 2.5 of solutions to, respectively, systems* (3) *and* (7)*, we impose constraints on the L^∞-norms of the, respectively, time marginal densities p_t and functions ρ_t. The L^∞ space cannot be an appropriate choice to search a global solution to the two-dimensional equation* (1) *under reasonable conditions on χ and ρ_0. Indeed, the singularity of the kernel $K_t^\sharp$ is stronger in dimension 2 than in dimension 1. Therefore, one can only construct local solutions to* (1) *and* (2) *when using the $L^\infty([0,T]\times\mathbb{R}^2)$-norm of the drift coefficient $B(t,x,p)$ and the $L^1([0,T];L^\infty(\mathbb{R}^2))$ norm of the time marginal density flow p. See [15, 14] for more comments on that issue and for the introduction of another technique of proof, based on accurate L^p-estimates, to construct global solutions under satisfying explicit conditions on χ and ρ_0.*

3 Preliminary: A density estimate

In the sequel, we will get local solutions to (3) and extend them to global solutions by means of an iterative procedure. The L^∞-norms of the successive drifts are needed to be bounded from above uniformly w.r.t. the iteration step. Standard density estimates obtained by using Girsanov theorem or PDE analysis do not help to this purpose. The reason is that they involve constants which exponentially depend on the L^∞-norm (or even Hölder-norm) of the drifts. We therefore proceed by using an accurate pointwise estimate (with explicit constants) on densities of one-dimensional diffusions with bounded measurable drifts. Estimate (11) below is obtained by using a stochastic technique. Its drawback is that the map $y \mapsto p_y^\beta(t,x,y)$ is not a probability density function. However, it suffices to nicely bound the successive drifts of the Picard iterations as shown by Proposition 5.3.

Let $X^{(b)}$ be a process defined by

$$X_t^{(b)} = X_0 + \int_0^t b(s, X_s^{(b)})\ ds + W_t, \quad t \in [0,T]. \tag{10}$$

To obtain $L^\infty(\mathbb{R})$ estimates for the transition probability density $p^{(b)}(t,x,y)$ of $X^{(b)}$ under the only assumption that the drift $b(t,x)$ is measurable and uniformly bounded we slightly extend the estimate proved in [13] for time homogeneous drift coefficients $b(x)$. We here propose a proof different from the original one. It avoids the use of densities of pinned diffusions and the claim that $p^{(b)}(t,x,y)$ is continuous w.r.t. all the variables which does not seem obvious to us. In our proof we adapt the method in [10], the main difference being that instead of the Wiener measure our reference measure is the probability distribution of the particular diffusion process X^β considered in [13] and defined by

$$X_t^\beta = X_0 + \beta \int_0^t \operatorname{sgn}(y - X_s^\beta)\ ds + W_t.$$

Theorem 3.1. *Let $X^{(b)}$ be the process defined in* (10) *with $X_0 = x$. Let $p_y^\beta(t,x,z)$ be the transition density of X^β. Assume $\beta := \sup_{t\in[0,T]} \|b(t,\cdot)\|_{L^\infty(\mathbb{R})} < \infty$. Then for all $y \in \mathbb{R}$ and $t \in (0,T]$ it holds that*

$$p^{(b)}(t,x,y) \le p_y^\beta(t,x,y) = \frac{1}{\sqrt{2\pi t}} \int_{\frac{|x-y|}{\sqrt{t}}}^\infty z e^{-\frac{(z-\beta\sqrt{t})^2}{2}}\, dz. \tag{11}$$

Proof. Let $f \in \mathcal{C}_c{}^\infty(\mathbb{R})$ and fix $t \in (0,T]$. Consider the parabolic PDE driven by the infinitesimal generator of X^β:

$$\begin{cases} \frac{\partial u}{\partial t}(s,x) + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(s,x) + \beta \operatorname{sgn}(y-x)\frac{\partial u}{\partial x}(s,x) = 0, \quad 0 \le s < t, \quad x \in \mathbb{R}, \\ u(t,x) = f(x), \quad x \in \mathbb{R}. \end{cases} \tag{12}$$

In view of [16, Thm.1] there exists a solution $u(s,x) \in W_p^{1,2}([0,t]\times R)$. Applying the Itô-Krylov formula to $u(s, X_s^\beta)$ we obtain that

$$u(s,x) = \int f(z) p_y^\beta(t-s,x,z)\ dz.$$

The formula (36) from our appendix allows us to differentiate under the integral sign:

$$\frac{\partial u}{\partial x}(s,x) = \int f(z) \frac{\partial p_y^\beta}{\partial x}(t-s,x,z)\ dz, \quad \forall 0 \le s < t \le T.$$

Fix $0 < \varepsilon < t$. Now apply the Itô-Krylov formula to $u(s, X_s^{(b)})$ for $0 \le s \le t-\varepsilon$ and use the PDE (12). It comes:

$$\mathbb{E}(u(t-\varepsilon, X_{t-\varepsilon}^{(b)})) = u(0,x) + \mathbb{E}\int_0^{t-\varepsilon} (b(s,X_s^{(b)}) - \beta \operatorname{sgn}(y - X_s^{(b)}))\frac{\partial u}{\partial x}(s, X_s^{(b)})\ ds.$$

In view of Corollary 8.2 in the appendix there exists a function $h \in L^1([0,t]\times\mathbb{R})$ such that

$$\forall 0 < s < t \le T,\ \forall y,z \in \mathbb{R}, \quad \mathbb{E}\left|\frac{\partial p_y^\beta}{\partial x}(t-s, X_s^{(b)}, z)\right| \le C_{T,\beta,x,y} h(s,z). \tag{13}$$

Consequently,

$$\mathbb{E}(u(t-\varepsilon, X_{t-\varepsilon}^{(b)})) = \int f(z) p_y^\beta(t,x,z)\; dz$$
$$+ \int f(z) \int_0^{t-\varepsilon} \mathbb{E}\left\{ (b(s,X_s^{(b)}) - \beta \text{sgn}(y - X_s^{(b)})) \frac{\partial p_y^\beta}{\partial x}(t-s, X_s^{(b)}, z) \right\} \; ds\; dz.$$

Let now ϵ tend to 0. By Lebesgue's dominated convergence theorem we obtain

$$\int f(z) p^{(b)}(t,x,z) dz = \int f(z) p_y^\beta(t,x,z) dz$$
$$+ \int f(z) \int_0^{t} \mathbb{E}\left\{ (b(s,X_s^{(b)}) - \beta \text{sgn}(y - X_s^{(b)})) \frac{\partial p_y^\beta}{\partial x}(t-s, X_s^{(b)}, z) \right\} \; ds\; dz.$$

Therefore the density $p^{(b)}$ satisfies:

$$p^{(b)}(t,x,z) = p_y^\beta(t,x,z) + \int_0^t \mathbb{E}\left\{ (b(s,X_s^{(b)}) - \beta \text{sgn}(y - X_s^{(b)})) \frac{\partial p_y^\beta}{\partial x}(t-s, X_s^{(b)}, z) \right\} \; ds.$$

As noticed in [13], in view of Formula (37) from our appendix we have for any $x \in \mathbb{R}$

$$(b(s,x) - \beta \text{sgn}(y-x)) \frac{\partial}{\partial x} p_y^\beta(t-s,x,y) \leq 0.$$

This leads us to choose $z = y$ in the preceding equality, which gives us

$$p^{(b)}(t,x,y) = p_y^\beta(t,x,y) + \int_0^t \mathbb{E}\left\{ (b(s,X_s^{(b)}) - \beta \text{sgn}(y - X_s^{(b)})) \frac{\partial p_y^\beta}{\partial x}(t-s, X_s^{(b)}, y) \right\} \; ds,$$

from which

$$\forall t \leq T, \quad p^{(b)}(t,x,y) \leq p_y^\beta(t,x,y).$$

We finally use Qian and Zheng's explicit representation (see [13] and our appendix section 8).

□

Corollary 3.2. *For $t > 0$ denote by $p(t, \cdot)$ the probability density of $X_t^{(b)}$. One has*

$$\|p(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2\pi t}} + \beta. \tag{14}$$

Proof. Denote by p_{X_0} the probability distribution X_0 in (10). In view of (11) we have

$$\begin{aligned}
p(t,y) &\leq \frac{1}{\sqrt{2\pi t}} \int p_{X_0}(dx) \int_{\frac{|x-y|}{\sqrt{t}}}^\infty z e^{-\frac{(z-\beta\sqrt{t})^2}{2}} dz dx \\
&\leq \frac{1}{\sqrt{2\pi t}} \int p_{X_0}(dx)(dx) \int_{\frac{|x-y|}{\sqrt{t}} - \beta\sqrt{t}}^\infty (z + \beta\sqrt{t}) e^{-\frac{z^2}{2}} dz dx \\
&= \frac{1}{\sqrt{2\pi t}} \Big(\int p_{X_0}(dx)(dx) e^{-\frac{(|x-y|-\beta t)^2}{2t}} dx + \beta\sqrt{t} \int p_{X_0}(dx)(dx) \int_{\frac{|x-y|}{\sqrt{t}} - \beta\sqrt{t}}^\infty e^{-\frac{z^2}{2}} dz dx \Big) \\
&\leq \frac{1}{\sqrt{2\pi t}} \int p_{X_0}(dx)(dx) e^{-\frac{(|y-x|-\beta t)^2}{2t}} dx + \beta.
\end{aligned}$$

□

Remark 3.3. *If the initial distribution has a density $p_0 \in L^\infty(\mathbb{R})$, the above calculation shows that*

$$\|p(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq 2\|p_0\|_{L^\infty(\mathbb{R})} + \beta.$$

If $p_0 \in L^p(\mathbb{R}), p > 1$, Hölder's inequality leads to

$$\frac{1}{\sqrt{2\pi t}} \int p_0(x) e^{-\frac{(|y-x|-\beta t)^2}{2t}} dx \leq \frac{\|p_0\|_{L^p(\mathbb{R})}}{\sqrt{2\pi t}} \big(\int e^{-q\frac{(|y-x|-\beta t)^2}{2t}} dx \big)^{1/q} \leq \frac{C_q t^{\frac{1}{2q}}}{\sqrt{t}} = \frac{C_q}{t^{\frac{1}{2p}}}.$$

4 A non-linear McKean–Vlasov–Fokker–Planck equation

This section is aimed to show that the time marginal densities of a weak solution to (3) uniquely solve a mild formulation of a McKean–Vlasov–Fokker–Planck equation. We will see in Section 7 that this mild equation reduces to (8) in the Keller–Segel case.

Proposition 4.1. *Let $T > 0$. Assume $b \in L^\infty([0,T] \times \mathbb{R})$ and Hypothesis (H). Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), X, W)$ be a weak solution to* (3) *until T. Then,*

1. *The time marginal densities $(p_t)_{t\in(0,T]}$ satisfy in the sense of the distributions the mild equation*

$$\forall t \in (0,T], \quad p_t = g_t * p_0 - \int_0^t \frac{\partial g_{t-s}}{\partial x} * (p_s(b(s,\cdot) + B(s,\cdot\,;p))ds. \tag{15}$$

2. *Equation* (15) *admits at most one solution $(p_t)_{t\in[0,T]}$ which for any $t \in (0,T]$ belongs to $L^1(\mathbb{R})$ and satisfies* (5).

Proof. We successively prove (15) and the uniqueness of its solution in $L^1(\mathbb{R})$.

1. Now, for $f \in C_b^2(\mathbb{R})$ consider the Cauchy problem

$$\begin{cases} \frac{\partial G}{\partial s} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2} = 0, \quad 0 \le s < t, \quad x \in \mathbb{R}, \\ \lim_{s\to t^-} G(s,x) = f(x). \end{cases} \tag{16}$$

The function

$$G_{t,f}(s,x) = \int f(y) g_{t-s}(x-y)dy$$

is a smooth solution to (16). Applying Itô's formula we get

$$\begin{aligned} G_{t,f}(t,X_t) - G_{t,f}(0,X_0) = &\int_0^t \frac{\partial G_{t,f}}{\partial s}(s,X_s)ds + \int_0^t \frac{\partial G_{t,f}}{\partial x}(s,X_s)(b(s,X_s) + B(s,X_s;p))ds \\ &+ \int_0^t \frac{\partial G_{t,f}}{\partial x}(s,X_s)dWs + \frac{1}{2}\int_0^t \frac{\partial^2 G_{t,f}}{\partial x^2}(s,X_s)ds. \end{aligned}$$

Using (16) we obtain

$$\mathbb{E}f(X_t) = \mathbb{E}G_{t,f}(0,X_0) + \int_0^t \mathbb{E}\left[\frac{\partial G_{t,f}}{\partial x}(s,X_s)(b(s,X_s) + B(s,X_s;p))\right]ds =: I + II. \tag{17}$$

On the one hand one has

$$I = \int\int f(y) g_t(y-x)dy \; p_0(dx) = \int f(y)(g_t * p_0)(y)dy.$$

On the second hand one has

$$\begin{aligned} II &= \int_0^t \int \frac{\partial}{\partial x}\big[\int f(y)g_{t-s}(x-y)dy\big](b(s,x) + B(s,x;p))p_s(x)dxds \\ &= \int_0^t \int\int f(y)\frac{\partial g_{t-s}}{\partial x}(x-y)dy(b(s,x) + B(s,x;p))p_s(x)dxds \\ &= -\int f(y)\int_0^t [\frac{\partial g_{t-s}}{\partial x} * ((b(s,\cdot) + B(s,\cdot;p))p_s)](y)dsdy. \end{aligned}$$

Thus (17) can be written as

$$\int f(y)p_t(y)dx = \int f(y)(g_t * p_0)(y)dy + \int f(y)\int_0^t [\frac{\partial g_{t-s}}{\partial x} * ((b(s,\cdot) + B(s,\cdot;p))p_s)](y)dsdy,$$

which is the mild equation (15).

2. Assume p_t^1 and p_t^2 are two mild solutions in the sense of the distributions to (15) which satisfy

$$\exists C_T > 0, \forall t \in (0,T], \quad \|p_t^1\|_{L^\infty(\mathbb{R})} + \|p_t^2\|_{L^\infty(\mathbb{R})} \leq \frac{C_T}{\sqrt{t}}.$$

Then, for every $t > 0$,

$$\begin{aligned}
\|p_t^1 - p_t^2\|_{L^1(\mathbb{R})} &\leq \int_0^t \|\frac{\partial g_{t-s}}{\partial x} * [B(s,\cdot\,;p^1)p_s^1 - B(s,\cdot\,;p^2)p_s^2)\|_{L^1(\mathbb{R})} ds \\
&\quad + \int_0^t \|\frac{\partial g_{t-s}}{\partial x} * [b(s,\cdot\,)(p_s^1 - p_s^2)]\|_{L^1(\mathbb{R})} ds \\
&\leq \int_0^t \|\frac{\partial g_{t-s}}{\partial x} * [(B(s,\cdot;p^1) - B(s,\cdot;p^2))p_s^1]\|_{L^1(\mathbb{R})} ds \\
&\quad + \int_0^t \|\frac{\partial g_{t-s}}{\partial x} * [(p_s^1 - p_s^2)B(s,\cdot\,;p^2)]\|_{L^1(\mathbb{R})} ds \\
&\quad + \int_0^t \|\frac{\partial g_{t-s}}{\partial x} * [b(s,\cdot\,)(p_s^1 - p_s^2)]\|_{L^1(\mathbb{R})} ds \\
&=: I + II + III.
\end{aligned}$$

As

$$\|\frac{\partial g_{t-s}}{\partial x}\|_{L^1(\mathbb{R})} \leq \frac{C_T}{\sqrt{t-s}},$$

the convolution inequality $\|f * h\|_{L^1(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})}\|h\|_{L^1(\mathbb{R})}$ and Remark 2.2 lead to

$$II \leq \int_0^t \|\frac{\partial g_{t-s}}{\partial x}\|_{L^1(\mathbb{R})}\|(p_s^1 - p_s^2)B(s,\cdot;p^2)\|_{L^1(\mathbb{R})} ds \leq C_T \int_0^t \frac{\|p_s^1 - p_s^2\|_{L^1(\mathbb{R})}}{\sqrt{t-s}} ds.$$

As b is bounded, we also have

$$|III| \leq C_T \int_0^t \frac{\|p_s^1 - p_s^2\|_{L^1(\mathbb{R})}}{\sqrt{t-s}} ds.$$

We now turn to I. Notice that

$$\|B(s,\cdot;p^1) - B(s,\cdot;p^2)\|_{L^1(\mathbb{R})} \leq \int_0^s \|K_{s-\tau}\|_{L^1(\mathbb{R})}\|p_\tau^1 - p_\tau^2\|_{L^1(\mathbb{R})} d\tau,$$

from which, since by hypothesis (p_t) satisfies (5),

$$\begin{aligned}
I &\leq \int_0^t \frac{C_T}{\sqrt{t-s}\sqrt{s}} \int_0^s \|K_{s-\tau}\|_{L^1(\mathbb{R})}\|p_\tau^1 - p_\tau^2\|_{L^1(\mathbb{R})} d\tau ds \\
&= \int_0^t \|p_\tau^1 - p_\tau^2\|_{L^1(\mathbb{R})} \int_\tau^t \frac{C_T}{\sqrt{t-s}\sqrt{s}} \|K_{s-\tau}\|_{L^1(\mathbb{R})} ds d\tau.
\end{aligned}$$

In addition, using Hypothesis (H-4),

$$\int_\tau^t \frac{1}{\sqrt{t-s}\sqrt{s}}\|K_{s-\tau}\|_{L^1(\mathbb{R})} ds \leq \frac{1}{\sqrt{\tau}} \int_\tau^t \frac{1}{\sqrt{t-s}}\|K_{s-\tau}\|_{L^1(\mathbb{R})} ds = \frac{1}{\sqrt{\tau}} \int_0^{t-\tau} \frac{\|K_s\|_{L^1(\mathbb{R})}}{\sqrt{t-\tau-s}} ds \leq \frac{C_T}{\sqrt{\tau}}.$$

It comes:

$$I \leq C_T \int_0^t \frac{\|p_\tau^1 - p_\tau^2\|_{L^1(\mathbb{R})}}{\sqrt{\tau}} d\tau.$$

Gathering the preceding estimates we obtain

$$\forall t > 0, \quad \|p_t^1 - p_t^2\|_{L^1(\mathbb{R})} \leq C_T \int_0^t \frac{\|p_s^1 - p_s^2\|_{L^1(\mathbb{R})}}{\sqrt{t-s}} ds + C_T \int_0^t \frac{\|p_s^1 - p_s^2\|_{L^1(\mathbb{R})}}{\sqrt{s}} ds.$$

Notice that $\lim_{t\to 0}\|p_t^1-p_t^2\|_{L^1(\mathbb{R})}=0$. Indeed, in view of (15) and Remark 2.2 one has

$$\|p_t^1-p_t^2\|_{L^1(\mathbb{R})}\leq\int_0^t\frac{C_T}{\sqrt{t-s}}(\|p_s^1\|_{L^1(\mathbb{R})}+\|p_s^2\|_{L^1(\mathbb{R})})ds\leq C_T\sqrt{t}.$$

Set $u(t):=\|p_t^1-p_t^2\|_{L^1(\mathbb{R})}$ for $t>0$ and $u(0)=0$. Applying the singular Gronwall Lemma 4.2 below to $u(t)$ we conclude

$$\forall t\in(0,T],\quad \|p_t^1-p_t^2\|_{L^1(\mathbb{R})}=0,$$

which ends the proof. □

In the above proof we have used the following result:

Lemma 4.2. *Let $(u(t))_{t\geq 0}$ be a non-negative bounded function such that for a given $T>0$, there exists a positive constant C_T such that for any $t\in(0,T]$:*

$$u(t)\leq C_T\int_0^t\frac{u(s)}{\sqrt{s}}ds+C_T\int_0^t\frac{u(s)}{\sqrt{t-s}}ds. \tag{18}$$

Then, $u(t)=0$ for any $t\in(0,T]$.

Proof. Inequality (18) leads to

$$u(t)\leq 2C_T\sqrt{t}\int_0^t\frac{u(s)}{\sqrt{s}\sqrt{t-s}}ds.$$

Iterate the preceding expression, apply Fubini's theorem and the definition of the β-function. It comes:

$$u(t)\leq(2C_T)^2\sqrt{t}\int_0^t\frac{\sqrt{s}}{\sqrt{s}\sqrt{t-s}}\int_0^s\frac{u(r)}{\sqrt{s-r}\sqrt{r}}dr\ ds\leq(2C_T)^2\sqrt{T}\beta(\frac{1}{2},\frac{1}{2})\int_0^t\frac{u(r)}{\sqrt{r}}dr.$$

It now remains to apply the classical Gronwall lemma. □

5 A local existence and uniqueness result for Equation (3)

Set

$$D(T):=\int_0^T\int_{\mathbb{R}}|K_t(x)|dxdt<\infty. \tag{19}$$

The main result in this section is the following theorem.

Theorem 5.1. *Let $T_0>0$ be such that $D(T_0)<1$. Assume $b\in L^\infty([0,T_0]\times\mathbb{R})$ continuous w.r.t. space variable. Under Hypothesis (H), Equation (3) admits a unique weak solution up to T_0 in the sense of Definition 2.1.*

Iterative procedure. Let us define the sequence $(X^k)_{k\geq 1}$ as follows. We start with

$$\begin{cases} dX_t^1=b(t,X_t^1)\ dt+\left\{\int_0^t(K_{t-s}*p_0)(X_t^1)ds\right\}dt+dW_t,\\ X_0^1\sim p_0.\end{cases} \tag{20}$$

Denote the drift of this equation by $b^1(t,x)$. For $k\geq 2$, suppose that, in the step $k-1$, the one dimensional time marginals of the law of the solution have densities $(p_t^{k-1})_{t>0}$, we define the drift in the step k as

$$b^k(t,x,p^{k-1})=b(t,x)+B(t,x;p^{k-1}).$$

The corresponding SDE is

$$\begin{cases} dX_t^k=b^k(t,X_t^k,p^{k-1})dt+dW_t,\\ X_0^k\sim p_0.\end{cases} \tag{21}$$

In order to prove the desired local existence and uniqueness result we set up the non-linear martingale problem related to (3).

Definition 5.2. *Consider the canonical space $\mathcal{C}([0,T_0];\mathbb{R})$ equipped with its canonical filtration. Let $\mathbb{Q}$ be a probability measure on this canonical space and denote by $\mathbb{Q}_t$ its one dimensional time marginals. $\mathbb{Q}$ solves the martingale problem $(MP(p_0,T_0,b))$ if*

(i) $\mathbb{Q}_0 = p_0$.

(ii) For any $t \in (0,T_0]$, $\mathbb{Q}_t$ have densities q_t w.r.t. Lebesgue measure on $\mathbb{R}$. In addition, they satisfy

$$\forall 0 < t \leq T_0, \quad \|q_t\|_{L^\infty(\mathbb{R})} \leq \frac{C_{T_0}}{\sqrt{t}}. \tag{22}$$

(iii) For any $f \in C_c^2(\mathbb{R})$ the process $(M_t)_{t\leq T_0}$, defined as

$$M_t := f(w_t) - f(w_0) - \int_0^t [\frac{1}{2}f''(w_u) + f'(w_u)(b(u,w_u) + \int_0^u \int K_{u-\tau}(w_u - y)q_\tau(y)dyd\tau)]du$$

is a $\mathbb{Q}$-martingale where (w_t) is the canonical process.

Notice that the arguments in Remark 2.2 justify that all the integrals in the definition of M_t are well defined.

We start with the analysis of Equations (20)-(21).

Proposition 5.3. *Same assumptions as in Theorem 5.1. For any $k \geq 1$, Equations (20)-(21) admit unique weak solutions up to T_0. For $k \geq 1$, denote by $\mathbb{P}^k$ the law of $(X_t^k)_{t\leq T_0}$. Moreover, for $t \in (0,T_0]$, the time marginals $\mathbb{P}_t^k$ of $\mathbb{P}^k$ have densities p_t^k w.r.t. Lebesgue measure on $\mathbb{R}$. Setting $\beta^k = \sup_{t\leq T_0} \|b^k(t,\cdot,p^{k-1})\|_{L^\infty(\mathbb{R})}$ and $b_0 := \|b\|_{L^\infty([0,T_0]\times\mathbb{R})}$, one has*

$$\forall 0 < t \leq T_0, \quad \|p_t^k\|_{L^\infty(\mathbb{R})} \leq \frac{C(b_0,T_0)}{\sqrt{t}} \quad \text{and} \quad \beta^k \leq C(b_0,T_0).$$

Finally, there exists a function $p^\infty \in L^\infty((0,T_0];L^1(\mathbb{R}))$ such that

$$\sup_{0<t\leq T_0} \|p_t^k - p_t^\infty\|_{L^1(\mathbb{R})} \to 0, \text{ as } k \to \infty.$$

Moreover,

$$\forall 0 < t \leq T_0, \quad \|p_t^\infty\|_{L^\infty(\mathbb{R})} \leq \frac{C(b_0,T_0)}{\sqrt{t}}. \tag{23}$$

Proof. We proceed by induction.

Case $k = 1$. In view of (H-5), one has $\beta^1 \leq b_0 + C_{T_0}$. This implies that the equation (20) has a unique weak solution in $[0,T_0]$ with time marginal densities $(p_t^1(y)dy)_{t\leq T_0}$ which in view of (14) satisfy

$$\forall t \in (0,T_0], \quad \|p_t^1\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2\pi t}} + \beta^1.$$

Case $k > 1$. Assume now that the equation for X^k has a unique weak solution and assume β^k is finite. In addition, suppose that the one dimensional time marginals satisfy

$$\forall t \in (0,T_0], \quad \|p_t^k\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\sqrt{2\pi t}} + \beta^k.$$

In view of (H-4), the new drift satisfies

$$|b^{k+1}(t,x;p^k)| \leq b_0 + \int_0^t \|p_s^k\|_{L^\infty(\mathbb{R})}\|K_{t-s}\|_{L^1(\mathbb{R})}ds \leq b_0 + \int_0^t (\frac{1}{\sqrt{2\pi s}} + \beta^k)\|K_{t-s}\|_{L^1(\mathbb{R})}ds \leq b_0 + C_{T_0} + \beta^k D(T_0).$$

Thus, we conclude that $\beta^{k+1} \leq b_0 + C_{T_0} + \beta^k D(T_0)$. Therefore, there exists a unique weak solution to the equation for X^{k+1}. Furthermore, by (14):

$$\forall t \in (0,T_0], \quad \|p_t^{k+1}\|_{L^\infty(\mathbb{R})} \leq \frac{C_{T_0}}{\sqrt{t}} + \beta^{k+1}.$$

Notice that

$$\forall k > 1, \quad \beta^{k+1} \leq b_0 + C_{T_0} + \beta^k D(T_0) \quad \text{and} \quad \beta^1 \leq b_0 + C_{T_0}.$$

Thus, as by hypothesis $D(T_0) < 1$, we have

$$\forall k \geq 1, \quad \beta^k \leq \frac{b_0 + C_{T_0}}{1 - D(T_0)} + b_0 + C_{T_0} \tag{24}$$

and

$$\|p_t^k\|_{L^\infty(\mathbb{R})} \leq \frac{C_{T_0}}{\sqrt{t}} + \beta^k \leq \frac{C_{T_0}}{\sqrt{t}} + \frac{b_0 + C_{T_0}}{1 - D(T_0)} + b_0 + C_{T_0} \leq \frac{C_{T_0}}{\sqrt{t}}. \tag{25}$$

Finally, it remains to prove that the sequence p^k converges in $L^\infty((0,T_0]; L^1(\mathbb{R}))$. In order to do so, we will prove p^k is a Cauchy sequence.

Applying the same procedure as in Section 4, one can derive the mild equation for $(p_t^k)_{t\in(0,T_0]}$. Thus, for every $k \geq 1$, the marginals $(p_t^k)_{t\in(0,T_0]}$ satisfy the mild equation

$$\forall t \in (0,T], \quad p_t^k = g_t * p_0 - \int_0^t \frac{\partial g_{t-s}}{\partial x} * (p_s^k b^k(s,\cdot,p^{k-1}))ds \tag{26}$$

in the sense of the distributions. Assume for a while that we have proved that for any $0 < t \leq T_0$, one has

$$\|p_t^k - p_t^{k-1}\|_{L^1(\mathbb{R})} \leq C_{T_0} \int_0^t \frac{\|p_s^{k-1} - p_s^{k-2}\|_{L^1(\mathbb{R})}}{\sqrt{s}} \, ds. \tag{27}$$

Remember that $\int_0^t f(u_1) \cdots \int_0^{u_{k-1}} f(u_k) du_k \; \dots du_1 = \frac{1}{k!}\left(\int_0^t f(u)du\right)^k$ for any positive integrable function f. Then, iterating (27) one gets,

$$\|p_t^k - p_t^{k-1}\|_{L^1(\mathbb{R})} \leq 2\frac{(C_{T_0}\sqrt{T_0})^{k-1}}{(k-1)!}.$$

Therefore, $\sup_{0<t\leq T_0} \|p_t^k - p_t^{k-1}\|_{L^1(\mathbb{R})} \to 0$, as $k \to \infty$ as desired.

It remains to prove the inequality (27). In view of (26), one has

$$\begin{aligned} \|p_t^k - p_t^{k-1}\|_{L^1(\mathbb{R})} &\leq \int_0^t \|\frac{\partial g_{t-s}}{\partial x} * (p_s^k b^k(s,\cdot,p^{k-1}) - p_s^{k-1} b^{k-1}(s,\cdot,p^{k-2}))\|_{L^1(\mathbb{R})} \, ds \\ &\leq \int_0^t \frac{1}{\sqrt{t-s}} \|b^{k-1}(s,\cdot,p^{k-2})(p_s^k - p_s^{k-1})\|_{L^1(\mathbb{R})} \, ds \\ &\quad + \int_0^t \frac{1}{\sqrt{t-s}} \|(b^k(s,\cdot,p^{k-1}) - b^{k-1}(s,\cdot,p^{k-2}))p_s^k\|_{L^1(\mathbb{R})} \, ds \\ &=: I + II. \end{aligned} \tag{28}$$

According to (24), one has

$$I \leq C_{T_0} \int_0^t \frac{\|p_s^k - p_s^{k-1}\|_{L^1(\mathbb{R})}}{\sqrt{t-s}} \, ds.$$

According to (25), one has

$$II \leq C_{T_0} \int_0^t \frac{1}{\sqrt{t-s}\sqrt{s}} \int_0^s \|K_{s-u} * (p_u^{k-1} - p_u^{k-2})\|_{L^1(\mathbb{R})} \, du \, ds.$$

Convolution inequality and Fubini-Tonelli's theorem lead to

$$II \leq C_{T_0} \int_0^t \|p_u^{k-1} - p_u^{k-2}\|_{L^1(\mathbb{R})} \int_u^t \frac{1}{\sqrt{t-s}\sqrt{s}} \|K_{s-u}\|_{L^1(\mathbb{R})} \, ds \, du.$$

Apply the change of variables $t - s = s'$. It comes,

$$II \leq C_{T_0} \int_0^t \frac{1}{\sqrt{u}} \|p_u^{k-1} - p_u^{k-2}\|_{L^1(\mathbb{R})} \int_0^{t-u} \frac{1}{\sqrt{s'}} \|K_{t-u-s'}\|_{L^1(\mathbb{R})} \, ds' \, du.$$

According to (H-4) one has

$$II \leq C_{T_0} \int_0^t \frac{1}{\sqrt{u}} \|p_u^{k-1} - p_u^{k-2}\|_{L^1(\mathbb{R})}\ du.$$

Coming back to (28) and using our above estimates on I and II, we obtain

$$\|p_t^k - p_t^{k-1}\|_{L^1(\mathbb{R})} \leq C_{T_0} \int_0^t \frac{\|p_s^k - p_s^{k-1}\|_{L^1(\mathbb{R})}}{\sqrt{t-s}}\ ds + C_{T_0} \int_0^t \frac{1}{\sqrt{u}} \|p_u^{k-1} - p_u^{k-2}\|_{L^1(\mathbb{R})}\ du.$$

We set $A_t := \int_0^t \frac{1}{\sqrt{u}} \|p_u^{k-1} - p_u^{k-2}\|_{L^1(\mathbb{R})}\ du$ and $\Phi(t) := \|p_t^k - p_t^{k-1}\|_{L^1(\mathbb{R})}$. Then, we have

$$\Phi(t) \leq C_{T_0} A(T_0) + C_{T_0} \int_0^t \frac{\Phi(s)}{\sqrt{t-s}}\ ds.$$

Iterating this relation, we get

$$\Phi(t) \leq C_{T_0} A(T_0) + C_{T_0}^2 \int_0^t \frac{1}{\sqrt{t-s}} \int_0^s \frac{\Phi(u)}{\sqrt{s-u}}\ du\ ds.$$

Apply Fubini's theorem to get

$$\Phi(t) \leq C_{T_0} A(T_0) + C_{T_0}^2 \int_0^t \Phi(u) \int_u^t \frac{1}{\sqrt{t-s}\sqrt{s-u}}\ ds\ du.$$

Notice that $\int_u^t \frac{1}{\sqrt{t-s}\sqrt{s-u}}\ ds = \int_0^1 \frac{1}{\sqrt{1-x}\sqrt{x}}\ dx$. Now, apply Gronwall's lemma to get (27) and the convergence of p^k to p^∞.

In order to obtain (23), fix $t \in (0, T]$ and use (25) and the fact that the convergence in $L^1(\mathbb{R})$ implies the almost sure convergence of a subsequence. □

The following is an obvious consequence of the preceding proposition:

Corollary 5.4. *Same assumptions as in Proposition 5.3. Assume that $(\mathbb{P}^k)_{k\geq 1}$ admits a weakly convergent subsequence $(\mathbb{P}^{n_k})_{k\geq 1}$. Denote its limit by $\mathbb{Q}$. Then for any $t \in (0, T_0]$, one has that $\mathbb{Q}_t(dx) = p_t^\infty(x)dx$, where p^∞ is constructed in Proposition 5.3.*

Proposition 5.5. *Same assumptions as in Theorem* (5.1)*. Then,*

1) The family of probabilities $(\mathbb{P}^k)_{k>1}$ is tight.

2) Any weak limit $\mathbb{P}^\infty$ of a convergent subsequence of $(\mathbb{P}^k)_{k\geq 1}$ solves $(MP(p_0, T_0, b))$.

Proof. In view of (24), we obviously have

$$\exists C_{T_0} > 0, \quad \sup_k \mathbb{E}|X_t^k - X_s^k|^4 \leq C_{T_0}|t-s|^2, \quad \forall\ 0 \leq s \leq t \leq T_0.$$

This is a sufficient condition for tightness (see e.g. [9, Chap.2, Pb.4.11]).

Let $(\mathbb{P}^{n_k})$ be a weakly convergent subsequence of $(\mathbb{P}^k)_{k\geq 1}$ and let $\mathbb{P}^\infty$ denote its limit. Let us check that $\mathbb{P}^\infty$ solves the martingale problem $(MP(p_0, T_0, b))$. To simplify the notation, we below write $\mathbb{P}^k$ instead of $\mathbb{P}^{n_k}$ and $\bar{p}^{k-1}$ instead of $p^{n_k - 1}$.

i) Each $\mathbb{P}_0^k$ has density p_0, and therefore $\mathbb{P}_0^\infty$ also has density p_0.

ii) Corollary 5.4 implies that the time marginals of $\mathbb{P}^\infty$ are absolutely continuous with respect to Lebesgue's measure and satisfy (22).

iii) Set

$$M_t := f(w_t) - f(w_0) - \int_0^t \big[\frac{1}{2} f''(w_u) + f'(w_u)(b(u, w_u) + \int_0^u (K_{u-\tau} * p_\tau^\infty)(w_u) d\tau)\big] du.$$

We have to prove

$$\mathbb{E}_{\mathbb{P}^\infty}[(M_t - M_s)\phi(w_{t_1}, \ldots, w_{t_N})] = 0, \quad \forall \phi \in C_b(\mathbb{R}^N) \text{ and } 0 \leq t_1 < \cdots < t_N < s \leq t \leq T_0, N \geq 1.$$

The process

$$M_t^k := f(w_t) - f(w_0) - \int_0^t \big[\frac{1}{2}f''(w_u) + f'(w_u)(b(u,w_u) + \int_0^u (K_{u-\tau} * \bar{p}_\tau^{k-1})(w_u)d\tau\big)\big]du$$

is a martingale under $\mathbb{P}^k$. Therefore, it follows that

$$\begin{aligned}
0 &= \mathbb{E}_{\mathbb{P}^k}[(M_t^k - M_s^k)\phi(w_{t_1},\dots,w_{t_N})] \\
&= \mathbb{E}_{\mathbb{P}^k}[\phi(\dots)(f(w_t) - f(w_s))] + \mathbb{E}_{\mathbb{P}^k}[\phi(\dots)\int_s^t \frac{1}{2}f''(w_u)du] \\
&+ \mathbb{E}_{\mathbb{P}^k}[\phi(\dots)\int_s^t f'(w_u)b(u,w_u)du] + \mathbb{E}_{\mathbb{P}^k}[\phi(\dots)\int_s^t f'(w_u)\int_0^u (K_{u-\tau} * \bar{p}_\tau^{k-1})(w_u)\ d\tau\ du].
\end{aligned}$$

Since $(\mathbb{P}^k)$ weakly converges to $\mathbb{P}^\infty$, the first two terms on the r.h.s. obviously converge. Now observe that

$$\begin{aligned}
&\mathbb{E}_{\mathbb{P}^k}[\phi(\cdots)\int_s^t f'(w_u)\int_0^u (K_{u-\tau} * \bar{p}_\tau^{k-1})(w_u)\ d\tau\ du] - \mathbb{E}_{\mathbb{P}^\infty}[\phi(\dots)\int_s^t f'(w_u)\int_0^u (K_{u-\tau} * p_\tau^\infty)(w_u)\ d\tau\ du] \\
&= \Big(\mathbb{E}_{\mathbb{P}^k}[\phi(\dots)\int_s^t f'(w_u)\int_0^u (K_{u-\tau} * \bar{p}_\tau^{k-1})(w_u)\ d\tau\ du] \\
&\quad - \mathbb{E}_{\mathbb{P}^k}[\phi(\dots)\int_s^t f'(w_u)\int_0^u (K_{u-\tau} * p_\tau^\infty)(w_u)\ d\tau\ du]\Big) \\
&\quad + \Big(\mathbb{E}_{\mathbb{P}^k}[\phi(\dots)\int_s^t f'(w_u)\int_0^u (K_{u-\tau} * p_\tau^\infty)(w_u)\ d\tau\ du] \\
&\quad - \mathbb{E}_{\mathbb{P}^\infty}[\phi(\dots)\int_s^t f'(w_u)\int_0^u (K_{u-\tau} * p_\tau^\infty)(w_u)\ d\tau\ du]\Big) \\
&=: I + II.
\end{aligned}$$

Now, in view of (25) and the definition of $D(T)$ as in (19), one has

$$\begin{aligned}
|I| &\le \|\phi\|_{L^\infty(\mathbb{R})}\int_s^t\int_0^u\int |f'(x)||(K_{u-\tau} * (\bar{p}_\tau^{k-1} - p_\tau^\infty))(x)|p_u^k(x)dx\ d\tau\ du \\
&\le \|\phi\|_{L^\infty(\mathbb{R})}\|f'\|_{L^\infty(\mathbb{R})}\int_s^t \frac{C_{T_0}}{\sqrt{u}}\int_0^u \|K_{u-\tau}\|_{L^1(\mathbb{R})}\|\bar{p}_\tau^{k-1} - p_\tau^\infty\|_{L^1(\mathbb{R})}d\tau\ du \\
&\le C_{T_0}D(T_0)\|\phi\|_{L^\infty(\mathbb{R})}\|f'\|_{L^\infty(\mathbb{R})}\sup_{r\le T_0}\|\bar{p}_r^{k-1} - p_r^\infty\|_{L^1(\mathbb{R})}.
\end{aligned}$$

Proposition 5.3 implies that $I \to 0$ as $k \to \infty$.

Now, to prove that $II \to 0$, it suffices to prove that the functional $F : \mathcal{C}([0,T_0];\mathbb{R}) \to \mathbb{R}$ defined by

$$w_\cdot \mapsto \phi(w_{t_1},\dots,w_{t_N})\int_s^t f'(w_u)\int_0^u\int K_{u-\tau}(w_u - y)p_\tau^\infty(y)\ dy\ d\tau\ du$$

is continuous. Let (w^n) a sequence converging in $\mathcal{C}([0,T_0];\mathbb{R})$ to w. Since ϕ is a continuous function, it suffices to show that

$$\begin{aligned}
&\lim_{n\to\infty}\int_s^t f'(w_u^n)\int_0^u\int K_{u-\tau}(w_u^n - y)p_\tau^\infty(y)\ dy\ d\tau\ du \\
&= \int_s^t f'(w_u)\int_0^u\int K_{u-\tau}(w_u - y)p_\tau^\infty(y)\ dy\ d\tau\ du.
\end{aligned} \tag{29}$$

For $(u,\tau) \in [s,t]\times[0,t]$, set

$$h_{u,\tau}(x) := \mathbb{1}\{\tau < u\}f'(x_u)\int K_{u-\tau}(x-y)p_\tau^\infty(y)dy.$$

The hypothesis (H-2) implies the continuity of $h_{u,\tau}$ on $\mathbb{R}$. Furthermore,

$$|h_{u,\tau}(x)| \leq C\mathbb{1}\{\tau < u\}\|p_\tau^\infty\|_{L^\infty(\mathbb{R})}\|K_{u-\tau}\|_{L^1(\mathbb{R})} \leq \frac{C}{\sqrt{\tau}}\mathbb{1}\{\tau < u\}\|K_{u-\tau}\|_{L^1(\mathbb{R})}.$$

In view of (H-4), we apply the theorem of dominated convergence to conclude (29). This ends the proof.

□

Proof of Theorem 5.1

Proposition 5.5 implies the existence of a weak solution $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), X, W)$ to (3) up to time T_0. Thus, the marginals $\mathbb{P} \circ X_t^{-1} =: p_t$ satisfy $\|p_t\|_{L^\infty(\mathbb{R})} \leq \frac{C}{\sqrt{t}}$, $t \in (0, T_0]$. In addition, as $|B(t, x; p)| \leq C(T_0)$, one has that $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), X, W)$ is the unique weak solution of the linear SDE

$$d\tilde{X}_t = b(t, \tilde{X}_t)dt + B(t, \tilde{X}_t; p)dt + dW_t, \quad t \leq T_0. \tag{30}$$

Now suppose that there exists another weak solution $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, (\hat{\mathcal{F}}_t), \hat{X}, \hat{W})$ to (3) up to T_0 and for $0 < t \leq T_0$ denote $\hat{\mathbb{P}} \circ \hat{X}_t^{-1}(dx) = \hat{p}_t(x)dx$. By Proposition 4.1 we have $\hat{p}_t = p_t$, for $0 < t \leq T_0$. Therefore, $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}, (\hat{\mathcal{F}}_t), \hat{X}, \hat{W})$ is a weak solution to (30), from which $\hat{\mathbb{P}} \circ \hat{X}^{-1} = \mathbb{P} \circ X^{-1}$.

6 Proofs of Theorem 2.3 and Corollary 2.4: A global existence and uniqueness result for Equation (3)

We now construct a solution for an arbitrary time horizon $T > 0$. We will do it by restarting the equation after the already fixed T_0. We start with $T = 2T_0$. Then, we will generalyze the procedure for an arbitrary $T > 0$.

6.1 An informal construction of the global weak solution on $[0, 2T_0]$

Assume, for a while, that a global weak solution uniquely exists. Denote the density of X_t by p_t^1, for $0 < t \leq T_0$ and by p_t^2, for $t \in (T_0, 2T_0]$. In view of Equation (3), we would have

$$X_{T_0+t} = X_{T_0} + \int_{T_0}^{T_0+t} b(s, X_s)ds + \int_{T_0}^{T_0+t}\int_0^s (K_{s-\theta} * p_\theta)(X_s)d\theta ds + W_{T_0+t} - W_{T_0}.$$

Observe that

$$\int_{T_0}^{T_0+t}\int_0^s (K_{s-\theta} * p_\theta)(X_s)d\theta ds = \int_{T_0}^{T_0+t}\int_0^{T_0} (K_{s-\theta} * p_\theta^1)(X_s)d\theta ds \ + \int_{T_0}^{T_0+t}\int_{T_0}^{s} (K_{s-\theta} * p_\theta^2)(X_s)dsdt$$
$$=: B_1 + B_2.$$

In addition,

$$B_1 = \int_0^t \int_0^{T_0} (K_{T_0+s'-\theta} * p_\theta^1)(X_{T_0+s'})d\theta ds',$$

and

$$B_2 = \int_0^t \int_{T_0}^{T_0+s'} (K_{T_0+s'-\theta} * p_\theta^2)(X_{T_0+s'})d\theta ds' = \int_0^t \int_0^{s'} (K_{s'-\theta'} * p_{T_0+\theta'}^2)(X_{T_0+s'})d\theta' ds'.$$

Now set $Y_t := X_{T_0+t}$ and $\tilde{p}_t(y) := p_{T_0+t}^2(y)$. Consider the new Brownian motion $\overline{W}_t := W_{T_0+t} - W_{T_0}$. It comes:

$$Y_t = Y_0 + \int_0^t b(s + T_0, Y_s)ds + \int_0^t \int_0^{T_0} (K_{T_0+s'-\theta} * p_\theta^1)(Y_s)d\theta ds + \int_0^t \int_0^s (K_{s'-\theta'} * \tilde{p}_\theta)(Y_s)d\theta ds + \overline{W}_t,$$

for $t \in (0, T_0]$. Setting

$$b_1(t, x, T_0) := \int_0^{T_0} (K_{T_0+t-s} * p_s^1)(x)ds \quad \text{and} \quad \tilde{b}(t, x) := b(T_0 + t, x),$$

we have

$$\begin{cases} dY_t = \tilde{b}(t, Y_t)dt + b_1(t, Y_t, T_0)dt + \Big\{ \int_0^t (K_{t-s} * \tilde{p}_s)(Y_t)ds \Big\} dt + d\overline{W}_t, \ t \le T_0, \\ Y_0 \sim p^1_{T_0}(y)dy, \ Y_s \sim \tilde{p}_s(y)dy. \end{cases} \tag{31}$$

The above calculation suggests the following procedure to construct a solution on $[0, 2T_0]$: One constructs a weak solution to (31) on $[0, T_0]$ and suitably paste its probability distribution with the solution to the non-linear martingale problem $(MP(p_0, T_0, b))$. We then prove that the so defined measure solves the desired non-linear martingale problem $(MP(p_0, 2T_0, b))$.

6.2 The global solution

Throughout this section, we denote by Ω_0 the canonical space $\mathcal{C}([0, T_0]; \mathbb{R})$ and by $\mathcal{B}_0$ its Borel σ- field. We denote by $\mathbb{Q}^1$ the probability distribution of the unique weak solution to (3) up to T_0 constructed in the previous section.

Lemma 6.1. *Let $T_0 > 0$ be such that $D(T_0) < 1$. Assume $b \in L^\infty([0, 2T_0] \times \mathbb{R})$ is continuous w.r.t. the space variable. Denote by p^1_t the time marginals of $\mathbb{Q}^1$. Set $b_1(t, x, T_0) := \int_0^{T_0} (K_{T_0+t-s} * p^1_s)(x)ds$ and $\tilde{b}(t, x) := b(T_0 + t, x)$. Under the hypothesis (H), Equation (31) admits a unique weak solution up to T_0.*

Proof. Let us check that we may apply Theorem 5.1 to (31). Notice that the SDE (31) involves the same kernel $K_t(x)$ as the SDE (3).

Firstly, by construction the initial law $p^1_{T_0}$ of Y satisfies the assumption of Theorem 5.1. Secondly, the role of the additional drift b is now played by the sum of the two linear drifts, $\tilde{b}$ and b_1. By hypothesis, $\tilde{b}$ is bounded in $[0, T_0] \times \mathbb{R}$ and continuous in the space variable. Using (5) and (H-6) we conclude that b_1 is bounded uniformly in t and x since

$$|b_1(t, x, T_0)| \le C_{T_0} \int_0^{T_0} \frac{\|K_{T_0+t-s}\|_{L^1(\mathbb{R})}}{\sqrt{s}} ds < C_{T_0}.$$

To show that $b_1(t, x, T_0)$ is continuous w.r.t. x we use (H-2) and proceed as at the end of the proof of Proposition 5.5.

We now are in a position to apply Theorem 5.1: There exists a unique weak solution to (31) up to T_0. □

Denote by $\mathbb{Q}^2$ the probability distribution of the process $(Y_t, t \le T_0)$. Notice that $\mathbb{Q}^2$ is the solution to the martingale problem $(MP(p^1_{T_0}, T_0, \tilde{b} + b_1))$.

A new measure on $\mathcal{C}([0, 2T_0]; \mathbb{R})$. Let $\mathbb{Q}^1$, $\mathbb{Q}^2$ and (p^1_t) be as above. Let (p^2_t) denote the time marginal densities of $\mathbb{Q}^2$. In particular, $\mathbb{Q}^2_0 = \mathbb{Q}^1_{T_0}$, i.e. $p^2_0(z)dz = p^1_{T_0}(z)dz$. Define the mapping X^0 from Ω_0 to $\mathbb{R}$ as $X^0(w) := w_0$. Using [9, Thm.3.19, Chap.5] to justify the introduction of regular conditional probabilities, for each $y \in \mathbb{R}$ we define the probability measure $\mathbb{Q}^2_y$ on $(\Omega_0, \mathcal{B}_0)$ by

$$\forall A \in \mathcal{B}_0, \quad \mathbb{Q}^2_y(A) = \mathbb{P}^2(A | X^0 = y).$$

In particular,

$$\mathbb{Q}^2_y(w \in \Omega_0, w_0 = y) = 1.$$

We now set $\Omega := \mathcal{C}([0, 2T_0]; \mathbb{R})$. For $w^1_\cdot, w^2_\cdot \in \Omega_0$ we define the concatenation $w = w^1 \otimes_{T_0} w^2 \in \Omega$ of these two paths as the function in Ω defined by

$$\begin{cases} w_\theta = w^1_\theta, & 0 \le \theta \le T_0, \\ w_{\theta+T_0} = w^1_{T_0} + w^2_\theta - w^2_0, & 0 \le \theta \le t - T_0. \end{cases}$$

On the other hand, for a given path $w \in \Omega$, the two paths $w^1_\cdot, w^2_\cdot \in \Omega_0$ such that $w = w^1 \otimes_{T_0} w^2$ are

$$\begin{cases} w^1_\theta = w_\theta, & 0 \le \theta \le T_0, \\ w^2_\theta = w_{T_0+\theta}, & 0 \le \theta \le T_0. \end{cases}$$

We define the probability distribution $\mathbb{Q}$ on Ω equipped with its Borel σ–field as follows. For any continuous and bounded functional φ on Ω,

$$\mathbb{E}_{\mathbb{Q}}[\varphi] = \int_{\Omega} \varphi(w)\ \mathbb{Q}(dw) := \int_{\Omega_0}\int_{\mathbb{R}}\int_{\Omega_0} \varphi(w^1 \otimes_{T_0} w^2)\ \mathbb{Q}^2_y(dw^2)\ p^1_{T_0}(y)dy\ \mathbb{Q}^1(dw^1). \tag{32}$$

Notice that if φ acts only on the part of the path up to $t \leq T_0$ of any $w. \in \Omega$, one has

$$\mathbb{E}_{\mathbb{Q}}[\varphi((w_\theta)_{\theta\leq t})] = \mathbb{E}_{\mathbb{Q}^1}[\varphi((w_\theta)_{\theta\leq t})]. \tag{33}$$

Proposition 6.2. *Let $T_0 > 0$ be such that $D(T_0) < 1$. Assume $b \in L^\infty([0, 2T_0] \times \mathbb{R})$ is continuous w.r.t. the space variable. Under the hypothesis (H), Equation* (3) *admits a unique weak solution up to $2T_0$.*

Proof. Let us prove that the probability measure $\mathbb{Q}$ solves the non–linear martingale problem $(MP(p_0, 2T_0, b))$ on the canonical space $\mathcal{C}([0, 2T_0]; \mathbb{R})$.

i) By (33), it is obvious that $\mathbb{Q}_0 = \mathbb{Q}^1_0$.

ii) Next, one easily obtains that the one dimensional time marginal densities of $\mathbb{Q}$ are p^1_t when $0 < t \leq T_0$ and $p^2_{t-T_0}$ when $T_0 < t \leq 2T_0$. Therefore, the one dimensional marginals of $\mathbb{Q}$ have densities q_t which, by construction, satisfy $\|q_t\|_{L^\infty(\mathbb{R})} \leq \frac{C}{\sqrt{t}}$ for $t \in (0, 2T_0]$.

iii) It remains to show that $(M_t)_{t\leq 2T_0}$ defined as

$$M_t := f(w_t) - f(w_0) - \int_0^t [\frac{1}{2}f''(w_u) + f'(w_u)(b(u, w_u) + \int_0^u \int K_{u-\tau}(w_u - y)q_\tau(y)dyd\tau\big)]du$$

is a $\mathbb{Q}$-martingale.

(a) For $s \leq t \leq T_0$ that results from (33) and the fact that $\mathbb{Q}^1$ solves the $(MP(p_0, T_0, b))$.

(b) For $s \leq T_0 \leq t \leq 2T_0$, in view of (a), it suffices to prove that $\mathbb{E}_{\mathbb{Q}}(M_t|\mathcal{B}_{T_0}) = M_{T_0}$. To prove the preceding holds true, use that $\mathbb{Q}^2$ solves $(MP(p^1_{T_0}, T_0, \tilde{b} + b_1))$ (see (c)).

(c) Let $T_0 \leq s \leq t \leq 2T_0$. For $w \in \Omega$ denote by w^1 and w^2 the functions in Ω_0 such that $w = w^1 \otimes_{T_0} w^2$. Proceeding as in the short calculation above Eq. (31) one easily gets

$$M_t - M_s = f(w^2_{t-T_0}) - f(w^2_{s-T_0}) - \int_{s-T_0}^{t-T_0} [\frac{1}{2}f''(w^2_u) - f'(w^2_u)(\tilde{b}(u, w^2_u)$$
$$+ b_1(u, w^2_u, T_0) + \int_0^u K_{u-\tau} * p^2_\tau(w^2_u)d\tau)]du =: F(w^2).$$

Now, take $t_1 < \cdots < t_m \leq T_0 < t_{m+1} < \cdots < t_N \leq s$ for $1 \leq m \leq N$.

$$\mathbb{E}_{\mathbb{Q}}(\phi(w_{t_1}, \ldots w_{t_N})(M_t - M_s)) =$$
$$\int_{\Omega_0}\int_{\mathbb{R}}\int_{\Omega_0} \phi(w^1_{t_1}, \ldots, w^1_{t_m}, w^2_{t_{m+1}-T_0}, \ldots, w^2_{t_N-T_0})F(w^2)\ \mathbb{Q}^2_y(dw^2)\ p^1_{T_0}(y)\ dy\ \mathbb{Q}^1(dw^1).$$

Since $\mathbb{Q}^2$ solves the $(MP(p^1_{T_0}, T_0, \tilde{b}+b_1))$, one has that $\mathbb{E}_{\mathbb{Q}^2}(\varphi(w^2_{t'_1}, \ldots w^2_{t'_n})F) = 0$ for any continuous bounded function φ on $\mathbb{R}^n$, any $n \in N$ and any $t'_1 \leq \cdots \leq t'_n < s - T_0$. Taking $\varphi(w^2_{t'_1}, \ldots w^2_{t'_{N-m}}) = \phi(w^1_{t_1}, \ldots, w^1_{t_m}, w^2_{t'_1}, \ldots, w^2_{t'_{N-m}})$ for a fixed w^1, we conclude that

$$\int_{\mathbb{R}}\int_{\Omega_0} \varphi(w^2_{t_{m+1}-T_0}, \ldots, w^2_{t_N-T_0})F(w^2)\mathbb{Q}^2_y(dw^2)p^1_{T_0}(y)dy = 0$$

which provides the desired result.

We have just shown the existence of a solution to $(MP(p_0, 2T_0, b))$. We proceed as in the proof of Theorem 5.1 to deduce the existence and uniqueness of a weak solution to (3) up to $2T_0$. $\square$

End of the proof of Theorem 2.3: From $2T_0$ to arbitrary time horizons. Given any finite time horizon $T > 0$, split the interval $[0, T]$ into $n = [\frac{T}{T_0}] + 1$ intervals of length not exceeding T_0 and repeat n times the procedure used in the preceding subsection. By construction, the time marginals of this solution to $(MP(p_0, T, b))$ have probability densities which satisfy (5). □

Let us now show that, under an additional assumption on the kernel K, the constraint (5) in Definition 2.1 may be relaxed without altering existence and uniqueness of the weak solution.

Proof of Corollary 2.4. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), X, W)$ be any solution to (3) in the sense of Definition 2.1. As (5) obviously implies (6) since p_t belongs to $L^1(\mathbb{R})$, $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), X, W)$ is a solution to (3) in the modified sense.

Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), X, W)$ be any solution to (3) in the new sense. Denote by $(p_t)_{0<t\leq T}$ the time marginal densities of $\mathbb{P} \circ X^{-1}$. Then, by Hölder's inequality, (6) and (H-7) imply that $B(t, x; p)$ is a bounded function on $[0, T] \times \mathbb{R}$. Therefore, in view of Corollary 3.2, $(p_t)_{0<t\leq T}$ satisfies (5) and, thus, $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), X, W)$ is a solution to (3) in the sense of Definition 2.1. □

Remark 6.3. *Another way to relax the condition* (5) *in Definition 2.1 is by adding regularity on the initial distribution. When this initial distribution has a density $p_0 \in L^\infty(\mathbb{R})$, we can use Remark 3.3 in the iterative procedure. Consequently, the constraint* (5) *may be relaxed as follows: the one dimensional marginal densities $(p_t)_{0<t\leq T}$ belong to $L^\infty([0, T] \times \mathbb{R})$. Similarly, if $p_0 \in L^p(\mathbb{R})$, then the constraint may be relaxed into*

$$\forall 0 < t \leq T, \quad \|p_t\|_{L^\infty(\mathbb{R})} \leq \frac{C_T}{t^{1/2p}}.$$

7 Application to the one-dimensional Keller–Segel model

In this section we prove Corollary 2.6. We start with checking that $K^\sharp$ satisfies Hypothesis (H). The condition (H-1) is satisfied since for $t > 0$ one has

$$\|K_t^\sharp\|_{L^1(\mathbb{R})} = \frac{C}{\sqrt{t}} \int |z| e^{-\frac{z^2}{2}} dz.$$

From the definition of $K^\sharp$ it is clear that for $t > 0$, $K_t^\sharp$ is a bounded and continuous function on $\mathbb{R}$. The condition (H-3) is also obviously satisfied.

To check (H-4), we notice

$$f_1(t) := \int_0^t \frac{\|K_{t-s}^\sharp\|_{L^1(\mathbb{R})}}{\sqrt{s}} ds = C \int_0^t \frac{1}{\sqrt{s}\sqrt{t-s}} ds = C \int_0^1 \frac{1}{\sqrt{x}\sqrt{1-x}} dx.$$

For $x \neq 0$, one has

$$\|K_\cdot^\sharp(x)\|_{L^1(0,T)} = C \int_0^T \frac{|x|}{s^{3/2}} e^{-\frac{|x|^2}{2s}} ds = C|x| \int_{\frac{|x|}{\sqrt{T}}}^\infty \frac{z^3}{|x|^3} e^{-\frac{z^2}{2}} \frac{|x|^2}{z^3} dz = C \int_{\frac{|x|}{\sqrt{T}}}^\infty e^{-\frac{z^2}{2}} dz \leq C.$$

For $x = 0$, one has $\|K_\cdot^\sharp(0)\|_{L^1(0,T)} = 0$. Thus, (H-5) is satisfied.

Finally, to prove (H-6) we notice that for every $t \in [0, T]$

$$\int_0^T \|K_{T+t-s}^\sharp\|_{L^1(\mathbb{R})} \frac{1}{\sqrt{s}} ds = \int_0^T \frac{C}{\sqrt{T+t-s}\sqrt{s}} ds \leq C \int_0^T \frac{1}{\sqrt{T-s}\sqrt{s}} ds = C.$$

Therefore, in view of Theorem 2.3, Equation (2) with $d = 1$ is well-posed.[1]

Denote by $\rho(t, x) \equiv p_t(x)$ the time marginals of the constructed probability distribution. Now, define the function c as in (9). In view of Inequality (5), for any $t \in (0, T]$ the function $c(t, \cdot)$ is well defined and bounded continuous. Let us show that $c \in L^\infty([0, T]; C_b^1(\mathbb{R}))$.

We have

$$\frac{\partial c}{\partial x}(t, x) = \frac{\partial}{\partial x} \left(e^{-\lambda t} \mathbb{E}(c_0(x + W_t))\right) + \frac{\partial}{\partial x} \left(\mathbb{E} \int_0^t e^{-\lambda s} \rho(t-s, x + W_s) ds\right).$$

[1] With similar calculations as for f_1, one easily checks that the function f_2 is bounded on any compact time interval. Thus, Corollary 2.4 applies as well as Theorem 2.3.

Then observe that

$$\begin{aligned}\frac{\partial}{\partial x}\mathbb{E}\int_0^t e^{-\lambda s}\rho(t-s,x+W_s)ds &= \frac{\partial}{\partial x}\int_0^t e^{-\lambda s}\int \rho(t-s,x+y)\frac{1}{\sqrt{2\pi s}}e^{\frac{-y^2}{4s}}dyds\\ &= \frac{\partial}{\partial x}\int_0^t e^{-\lambda(t-s)}\int \rho(s,y)\frac{1}{\sqrt{2\pi(t-s)}}e^{\frac{-(y-x)^2}{4(t-s)}}dyds\\ &=: \frac{\partial}{\partial x}\int_0^t f(s,x)ds.\end{aligned}$$

As for any $0<s<t$

$$|\frac{\partial}{\partial x}\frac{1}{\sqrt{t-s}}e^{\frac{-(y-x)^2}{2(t-s)}}| \le \frac{|y-x|}{2(t-s)^{3/2}}e^{\frac{-(y-x)^2}{2(t-s)}} \le \frac{C}{t-s},$$

we have

$$f'(s,x) = e^{-\lambda(t-s)}\int \rho(s,y)\frac{y-x}{2\sqrt{2\pi}(t-s)^{3/2}}e^{\frac{-(y-x)^2}{2(t-s)}}dy.$$

Now, we repeat the same argument for $\frac{\partial}{\partial x}\int_0^t f(s,x)ds$. In order to justify the differentiation under the integral sign we notice that

$$|f'(s,x)| \le \frac{C_T}{\sqrt{(t-s)s}}.$$

Gathering the preceding calculations we have obtained

$$\frac{\partial c}{\partial x}(t,x) = e^{-\lambda t}\mathbb{E}c_0'(x+W_t) + \int_0^t e^{-\lambda(t-s)}\int \rho_s(y)\frac{y-x}{\sqrt{2\pi}(t-s)^{3/2}}e^{\frac{-(y-x)^2}{2(t-s)}}dy\ ds. \tag{34}$$

Using the assumption on c_0 and Inequality (5), for any $t\in(0,T]$ one has

$$\|\frac{\partial c}{\partial x}(t,\cdot)\|_{L^\infty(\mathbb{R})} \le \|c_0'\|_{L^\infty(\mathbb{R})} + C_T.$$

In addition, the preceding calculation and Lebesgue's Dominated Convergence Theorem show that $\frac{\partial c}{\partial x}(t,\cdot)$ is continuous on $\mathbb{R}$. We thus have obtained as desired that $c\in L^\infty([0,T];C_b^1(\mathbb{R}))$.

The above discussion shows that we are in a position to apply Proposition 4.1 with $b(t,x)\equiv \chi e^{-\lambda t}\mathbb{E}c_0'(x+W_t)$ and $B(t,x;\rho)$ defined as in (4) with $K\equiv K^\sharp$: In view of (34) one has $b(t,x)+B(t,x;\rho)=\frac{\partial c}{\partial x}(t,x)$ and therefore the function $\rho(t,x)$ satisfies (8) in the sense of the distributions. Therefore, it is a solution to the Keller Segel system (7) in the sense of Definition 2.5. We now check the uniqueness of this solution.

Assume there exists another solution ρ^1 satisfying Definition 2.5 with the same initial condition as ρ. For notation convenience, in the calculation below we set $c_t(x):=c(t,x)$, $c_t^1(x):=c^1(t,x)$, $\rho_t(x):=\rho(t,x)$, and $\rho_t^1(x):=\rho^1(t,x)$.

Using Definition 2.5, for every $t>0$

$$\begin{aligned}&\|\rho_t^1-\rho_t\|_{L^1(\mathbb{R})} \le \int_0^t \|\frac{\partial g_{t-s}}{\partial x}*(\frac{\partial c_s^1}{\partial x}\rho_s^1 - \frac{\partial c_s}{\partial x}\rho_s)\|_{L^1(\mathbb{R})}ds\\ &\le \int_0^t \|\frac{\partial g_{t-s}}{\partial x}*(\frac{\partial c^1}{\partial x}(\rho_s^1-\rho_s))\|_{L^1(\mathbb{R})}ds + \int_0^t \|\frac{\partial g_{t-s}}{\partial x}*(\rho_s(\frac{\partial c^1}{\partial x}-\frac{\partial c_s}{\partial x}))\|_{L^1(\mathbb{R})}ds\\ &=: I+II.\end{aligned}$$

Using standard convolution inequalities and $\|\frac{\partial g_{t-s}}{\partial x}\|_{L^1(\mathbb{R})}\le \frac{C}{\sqrt{t-s}}$ we deduce:

$$I \le C\int_0^t \frac{\|\rho_s^1-\rho_s\|_{L^1(\mathbb{R})}}{\sqrt{t-s}}ds \quad \text{and} \quad II \le C\int_0^t \frac{\|\frac{\partial c_s^1}{\partial x}-\frac{\partial c_s}{\partial x}\|_{L^1(\mathbb{R})}}{\sqrt{t-s}\sqrt{s}}ds.$$

Therefore

$$\|\frac{\partial c_s^1}{\partial x}-\frac{\partial c_s}{\partial x}\|_{L^1(\mathbb{R})} \le \int_0^s \|(\rho_u^1-\rho_u)*\frac{\partial g_{s-u}}{\partial x}\|_{L^1(\mathbb{R})}du \le C\int_0^s \frac{\|\rho_u^1-\rho_u\|_{L^1(\mathbb{R})}}{\sqrt{s-u}}du, \tag{35}$$

from which

$$II \le C \int_0^t \frac{1}{\sqrt{s}\sqrt{t-s}} \int_0^s \frac{\|\rho_u^1 - \rho_u\|_{L^1(\mathbb{R})}}{\sqrt{s-u}}\ du\ ds$$
$$\le C \int_0^t \|\rho_u^1 - \rho_u\|_{L^1(\mathbb{R})} \int_u^t \frac{1}{\sqrt{s}\sqrt{s-u}\sqrt{t-s}}\ ds\ du \le C_T \int_0^t \frac{\|\rho_u^1 - \rho_u\|_{L^1(\mathbb{R})}}{\sqrt{u}} du.$$

Gathering the preceding bounds for I and II we get

$$\|\rho_t^1 - \rho_t\|_{L^1(\mathbb{R})} \le C_T \int_0^t \frac{\|\rho_s^1 - \rho_s\|_{L^1(\mathbb{R})}}{\sqrt{t-s}} ds + C_T \int_0^t \frac{\|\rho_s^1 - \rho_s\|_{L^1(\mathbb{R})}}{\sqrt{s}} ds.$$

Similarly as at the end of the proof of Proposition 4.1 we notice that $\lim_{t\to 0} \|\rho_t^1 - \rho_t^2\|_{L^1(\mathbb{R})} = 0$. Lemma 4.2 thus implies that $\|\rho_t^1 - \rho_t\|_{L^1(\mathbb{R})} = 0$ for every $0 < t \le T$. In view of (35) we also have $\|\frac{\partial c_t^1}{\partial x} - \frac{\partial c_t}{\partial x}\|_{L^1(\mathbb{R})} = 0$. This completes the proof of Corollary 2.6.

8 Appendix

We here propose a light simplification of the calculations in [13].

Proposition 8.1. *Let $y \in \mathbb{R}$ and let β be a constant. Denote by $p_y^\beta(t,x,z)$ the transition probability density (with respect to the Lebesgue measure) of the unique weak solution to*

$$X_t = x + \beta \int_0^t sgn(y - X_s)\ ds + W_t.$$

Then

$$p_y^\beta(t,x,z) = \frac{1}{\sqrt{2\pi} t^{3/2}} \int_0^\infty e^{\beta(|y-x|+\bar{y}-|z-y|) - \frac{\beta^2}{2} t} (\bar{y} + |z-y| + |y-x|) e^{-\frac{(\bar{y}+|z-y|+|y-x|)^2}{2t}} d\bar{y}$$
$$+ \frac{1}{\sqrt{2\pi t}} e^{\beta(|y-x|-|z-y|) - \frac{\beta^2}{2} t} (e^{-\frac{(z-x)^2}{2t}} - e^{-\frac{(|z-y|+|y-x|)^2}{2t}}). \tag{36}$$

In particular,

$$p_y^\beta(t,x,y) = \frac{1}{\sqrt{2\pi t}} \int_{\frac{|x-y|}{\sqrt{t}}}^\infty z e^{-\frac{(z-\beta\sqrt{t})^2}{2}}\ dz. \tag{37}$$

Proof. Let f be a bounded continuous function. The Girsanov transform leads to

$$\mathbb{E}(f(X_t)) = \mathbb{E}(f(x+W_t) e^{\beta \int_0^t \operatorname{sgn}(y-x-W_s) dW_s - \frac{\beta^2}{2} t}).$$

Let L_t^a be the Brownian local time. By Tanaka's formula ([9], p. 205):

$$|W_t - a| = |a| + \int_0^t \operatorname{sgn}(W_s - a) dW_s + L_t^a.$$

Therefore

$$\int_0^t \operatorname{sgn}(y - x - W_s) dW_s = |y-x| + L_t^a - |W_t - (y-x)|,$$

from which

$$\mathbb{E}(f(X_t)) = \mathbb{E}(f(x+W_t) e^{\beta(|y-x| + L_t^{y-x} - |W_t - (y-x)|) - \frac{\beta^2}{2} t}).$$

Recall that (W_t, L_t^a) has the following joint distribution:

$$\begin{cases} \bar{y} > 0: \quad \mathbb{P}(W_t \in dz, L_t^a \in d\bar{y}) = \frac{1}{\sqrt{2\pi} t^{3/2}} (\bar{y} + |z-a| + |a|) e^{-\frac{(\bar{y}+|z-a|+|a|)^2}{2t}} d\bar{y} dz. \\ \mathbb{P}(W_t \in dz, L_t^a = 0) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{z^2}{2t}} dz - \frac{1}{\sqrt{2\pi t}} e^{-\frac{(|z-a|+|a|)^2}{2t}} dz \end{cases}$$

(see [1, p. 200, Eq. (1.3.8)]). It comes:

$$\mathbb{E}(f(X_t)) = \frac{1}{\sqrt{2\pi}t^{3/2}} \int_{\mathbb{R}} \int_0^\infty f(x+z) e^{\beta(|y-x|+\bar{y}-|z-(y-x)|)-\frac{\beta^2}{2}t} (\bar{y}+|z-(y-x)|+|y-x|) \\ e^{-\frac{(\bar{y}+|z-(y-x)|+|y-x|)^2}{2t}} \, d\bar{y} \, dz \\ + \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(x+z) e^{\beta(|y-x|-|z-(y-x)|)-\frac{\beta^2}{2}t} (e^{-\frac{z^2}{2t}} - e^{-\frac{(|z-(y-x)|+|y-x|)^2}{2t}}) \, dz.$$

The change of variables $x+z=z'$ leads to

$$\mathbb{E}(f(X_t)) = \frac{1}{\sqrt{2\pi}t^{3/2}} \int_{\mathbb{R}} f(z') \int_0^\infty e^{\beta(|y-x|+\bar{y}-|z'-y|)-\frac{\beta^2}{2}t} (\bar{y}+|z'-y)|+|y-x|) e^{-\frac{(\bar{y}+|z'-y|+|y-x|)^2}{2t}} \, d\bar{y} dz' \\ + \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(z') e^{\beta(|y-x|-|z'-y|)-\frac{\beta^2}{2}t} (e^{-\frac{(z'-x)^2}{2t}} - e^{-\frac{(|z'-y|+|y-x|)^2}{2t}}) dz',$$

from which the desired result follows.

□

In the next Corollary we use the same notation as in the proof of Theorem 3.1.

Corollary 8.2. *Let $0 < s < t \le T$. Then for any $z, y \in \mathbb{R}$, there exists $C_{T,\beta,x,y}$ such that*

$$\mathbb{E}| \left(\frac{\partial}{\partial x} p_y^\beta \right) (t-s, X_s^{(b)}, z)| \le C_{T,\beta,x,y} h(s,z),$$

where h belongs to $L^1([0,t] \times \mathbb{R})$.

Proof. By Girsanov's theorem, for some constant $C_{T,\beta}$ we have

$$\mathbb{E} \left| \left(\frac{\partial}{\partial x} p_y^\beta \right) (t-s, X_s^{(b)}, z) \right| \le C_{T,\beta} \sqrt{\mathbb{E} \left| \left(\frac{\partial}{\partial x} p_y^\beta \right) (t-s, W_s^x, z) \right|^2}.$$

Observe that

$$\frac{\partial}{\partial \bar{x}} p_y^\beta (t-s, \bar{x}, z) = \frac{\beta}{\sqrt{2\pi(t-s)}} e^{-2\beta|z-y|} e^{-\frac{(|z-y|+|y-\bar{x}|-\beta(t-s))^2}{2(t-s)}} \operatorname{sgn}(\bar{x}-y) \\ + \frac{\beta}{\sqrt{2\pi(t-s)}} e^{-\beta|z-y|-\frac{\beta^2}{2}(t-s)} e^{\beta|y-\bar{x}|-\frac{(z-\bar{x})^2}{2(t-s)}} \operatorname{sgn}(\bar{x}-y) \\ + \frac{z-\bar{x}}{2\pi(t-s)^{3/2}} e^{-\beta|z-y|-\frac{\beta^2}{2}(t-s)} e^{\beta|y-\bar{x}|-\frac{(z-\bar{x})^2}{2(t-s)}}.$$

The sum of the absolute values of the first two terms in the right-hand side is bounded from above by

$$\frac{\beta}{\sqrt{2\pi(t-s)}} e^{-2\beta|z-y|+\beta|y-\bar{x}|}.$$

Thus,

$$\mathbb{E} \left| \left(\frac{\partial}{\partial x} p_y^\beta \right) (t-s, X_s^{(b)}, z) \right| \le \frac{C_{T,\beta}}{\sqrt{2\pi(t-s)}} \sqrt{\mathbb{E} e^{2\beta|y-W_s^x|}} + \frac{C_{T,\beta}}{(t-s)^{3/2}} \sqrt{\mathbb{E}(|z-W_s^x|^2 e^{2\beta|y-W_s^x|-\frac{(z-W_s^x)^2}{t-s}})} \\ =: B + A.$$

Notice that

$$A \le \frac{C_{T,\beta}}{(t-s)^{3/2}} (\mathbb{E}[|z-W_s^x|^4 e^{-2\frac{(z-W_s^x)^2}{t-s}}] \mathbb{E}[e^{4\beta|y-W_s^x|}])^{1/4} =: \frac{C_{T,\beta}}{(t-s)^{3/2}} (A_1 A_2)^{1/4}.$$

Firstly, as there exists an $\alpha > 0$ such that $|a|^4 e^{-a^2} \le C e^{-\alpha a^2}$, one has

$$A_1 \le C(t-s)^2 \int e^{-\alpha \frac{(z-u)^2}{t-s}} g_s(u-x) du \le \frac{(t-s)^{2+1/2}}{\sqrt{s+(t-s)/(2\alpha)}} e^{-\frac{(z-x)^2}{2(s+(t-s)/(2\alpha))}}.$$

Secondly,

$$A_2 = \int e^{4\beta|y-u|} g_s(u-x)du = e^{-4\beta y} \int_y^\infty e^{4\beta u} \frac{1}{\sqrt{s}} e^{-\frac{(u-x)^2}{2s}} du + e^{4\beta y} \int_{-\infty}^y e^{-4\beta u} \frac{1}{\sqrt{s}} e^{-\frac{(u-x)^2}{2s}} du$$
$$= e^{4\beta(x-y)} e^{8\beta^2 s} \int_y^\infty \frac{1}{\sqrt{s}} e^{-\frac{(u-x-4\beta s)^2}{2s}} du + e^{4\beta(y-x)} e^{8\beta^2 s} \int_{-\infty}^y \frac{1}{\sqrt{s}} e^{-\frac{(u-x+4\beta s)^2}{2s}} du \le e^{8\beta^2 s} C_{\beta,x,y}.$$

Therefore,

$$A \le C_{T,\beta,x,y} \frac{1}{(t-s)^{7/8}} g_{s+(t-s)/(2\alpha)}(z-x).$$

The term B is treated in the similar way as A_2. $\square$

References

[1] Borodin, A. N., and Salminen, P. *Handbook of Brownian motion—facts and formulae.* Probability and its Applications. Birkhäuser Verlag, Basel, 1996.

[2] Carrapatoso, K., and Mischler, S. Uniqueness and long time asymptotics for the parabolic–parabolic keller–segel equation. *Communications in Partial Differential Equations 42*, 2 (2017), 291–345.

[3] Cattiaux, P., and Pédèches, L. The 2-D stochastic Keller-Segel particle model: existence and uniqueness. *ALEA Lat. Am. J. Probab. Math. Stat. 13*, 1 (2016), 447–463.

[4] Corrias, L., Escobedo, M., and Matos, J. Existence, uniqueness and asymptotic behavior of the solutions to the fully parabolic Keller-Segel system in the plane. *J. Differential Equations 257*, 6 (2014), 1840–1878.

[5] Fournier, N., and Jourdain, B. Stochastic particle approximation of the Keller–Segel equation and two-dimensional generalization of Bessel processes. *Ann. Appl. Probab. 27*, 5 (2017), 2807–2861.

[6] Haškovec, J., and Schmeiser, C. Convergence of a stochastic particle approximation for measure solutions of the 2D Keller-Segel system. *Comm. Partial Differential Equations 36*, 6 (2011), 940–960.

[7] Hillen, T., and Potapov, A. The one-dimensional chemotaxis model: global existence and asymptotic profile. *Math. Methods Appl. Sci. 27*, 15 (2004), 1783–1801.

[8] Jabir, J.-F., Talay, D., and Tomašević, M. Mean–field limit of a particle approximation of the one-dimensional parabolic–parabolic Keller-Segel model without smoothing. *Electronic Communications in probability 23*, 84 (2018), 1–14.

[9] Karatzas, I., and Shreve, S. E. *Brownian motion and stochastic calculus*, second ed., vol. 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991.

[10] Makhlouf, A. Representation and Gaussian bounds for the density of Brownian motion with random drift. *Commun. Stoch. Anal. 10*, 2 (2016), 151–162.

[11] Osaki, K., and Yagi, A. Finite dimensional attractor for one-dimensional Keller-Segel equations. *Funkcial. Ekvac. 44*, 3 (2001), 441–469.

[12] Perthame, B. PDE models for chemotactic movements: parabolic, hyperbolic and kinetic. *Appl. Math. 49*, 6 (2004), 539–564.

[13] Qian, Z., and Zheng, W. Sharp bounds for transition probability densities of a class of diffusions. *C. R. Math. Acad. Sci. Paris 335*, 11 (2002), 953–957.

[14] Tomasevic, M. A new McKean-Vlasov stochastic interpretation of the parabolic-parabolic Keller-Segel model: The two-dimensional case. *Preprint, hal: 02044262*.

[15] Tomašević, M. Ph.d. thesis.

[16] Veretennikov, A. Y. Parabolic equations and itô's stochastic equations with coefficients discontinuous in the time variable. *Mathematical notes of the Academy of Sciences of the USSR 31*, 4 (1982), 278–283.